heat conductivity of the medium; σ, Stefan — Boltzmann constant; T, temperature, °K; τ_0^0, optical thickness for unit porosity.

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THERMAL INSTABILITY OF A VISCOELASTIC FLUID LAYER WITH LIFT AND THERMOCAPILLARY FORCES TAKEN INTO ACCOUNT

F. A. Garifullin

UDC 532.135

The stability problem of a viscoelastic fluid layer of integral type is investigated by the Fourier method during heating from below. The simultaneous effect of the lift and thermocapillary forces is taken into account. The critical values of the Rayleigh and Marangoni criteria are determined.

The stability of a horizontal viscoelastic fluid layer heated from below has been considered up to now only under the effect of Archimedes forces [1, 2]. However, another instability mechanism is possible — the change in the thermocapillary forces on the free fluid surface [3]. In the general case, instability can originate as a result of the simultaneous action of these two forces.

Let us consider an infinite horizontal viscoelastic fluid layer bounded from above by an undeformable free surface and from below by a solid mass of finite thickness and heat conductivity (Fig. 1). The surface $z = -d_1$ is maintained at the constant temperature T_1^0, and heat is transmitted from the free surface $z = d$ to the surrounding medium with temperature T_2^0 by convection.

The thermal boundary conditions of this problem can be formulated in the form

$$T = T_1^0 \quad \text{for} \quad z = -d_1, \tag{1}$$

$$T = T_1,\ \varkappa_1 \frac{\partial T_1}{\partial z} = \varkappa \frac{\partial T}{\partial z} \quad \text{for} \quad z = 0, \tag{2}$$

$$\alpha (T - T_2^0) = -\varkappa \frac{\partial T}{\partial z} \quad \text{for} \quad z = d. \tag{3}$$

The amplitude equations of the perturbed state are written in the Boussinesq approximation in the form

$$[\sigma \,\mathrm{Pr}^{-1} - \psi(\sigma)(D^2 - \gamma^2)](D^2 - \gamma^2) W = -R\gamma^2\Theta, \tag{4}$$

$$(\sigma - D^2 + \gamma^2)\Theta = W, \tag{5}$$

$$(D^2 - \gamma^2 - \sigma/\bar{\varkappa})\Theta_1 = 0. \tag{6}$$

The previous dimensionless variables [2] were hence used.

The boundary conditions for the perturbations are expressed by the following dependences:

$$\Theta(0) = \Theta_1(0),\ D\Theta(0) = \bar{\varkappa} D\Theta_1(0), \tag{7}$$

$$\Theta_1(-L) = 0, \quad L = d_1/d, \tag{8}$$

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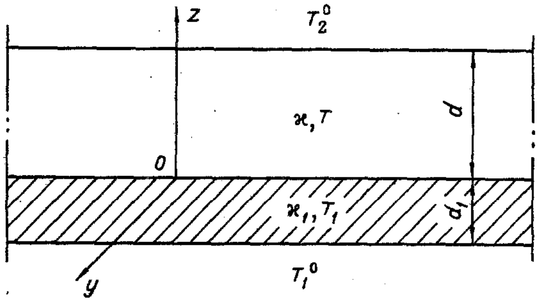

Fig. 1. Fluid heating diagram.

TABLE 1. Values of the Critical Parameters for Monotonic Instability

B	A=0				A=∞			
	M_*	γ_*^M	R_*	γ_*^R	M_*	γ_*^M	R_*	γ_*^M
0	48,13	0,0	32,12	0,0	79,82	1,99	669,2	2,076
0,1	58,31	1,06	38,57	1,03	83,66	2,03	684,4	2,12
1,0	96,70	1,71	517,8	1,65	116,56	2,25	772,7	2,35
10	385,4	2,42	727,1	2,11	416,13	2,74	988,5	2,63
100	3153,8	2,69	808,8	2,25	3307,6	2,97	1084,3	2,68
1000	30797,8	2,73	817,2	2,27	32441,3	3,00	1099,2	2,69

$$W(0) = DW(0) = W(1) = 0, \tag{9}$$

where the conditions

$$D\Theta(1) = -B\Theta(1), \; D^2W(1) = -M\gamma^2\Theta(1)(1+\sigma L) \tag{10}$$

exist on the surface. The solution of (6) that satisfies condition (8) is

$$\Theta_1(z) = C\,\mathrm{sh}\,[(\gamma^2 + \sigma/\bar{\varkappa})^{1/2}(L+z)], \tag{11}$$

where C is an arbitrary constant. Condition (7) permits writing

$$D\Theta(0) = A\Theta(0), \tag{12}$$

where

$$A = \bar{\varkappa}(\gamma^2 + \sigma/\bar{\varkappa})^{1/2}\,\mathrm{cth}\,[L(\gamma^2 + \sigma/\bar{\varkappa})^{1/2}].$$

The thermal boundary conditions (12) include the two limit cases

$$A = 0, \; (D\Theta(0) = 0), \; A = \infty, \; (\Theta(0) = 0).$$

The first case corresponds to a boundary with perfect heat insulation ($\varkappa \ll 1$) and the second to perfectly heat conducting boundaries ($\varkappa \gg 1$).

To solve the problem, let us use the Fourier-series expansion method. Following the procedure described in [4-7], the perturbation amplitudes can be approximated by the expressions

$$W(z) = \sum_{n=1}^{\infty}\left[A_n + (-1)^n\frac{2}{(n\pi)^3}D^2W(1) - \frac{2}{(n\pi)^3}D^2W(0) + \right.$$
$$\left. + \frac{2}{n\pi}(-1)^{n+1}W(1) + \frac{2}{n\pi}W(0)\right]\sin n\pi z, \tag{13}$$

$$\Theta(z) = \sum_{n=1}^{\infty}\left[B_n - (-1)^n\frac{2}{n\pi}\Theta(1) + \frac{2}{n\pi}\Theta(0)\right]\sin n\pi z. \tag{14}$$

Use of these dependences in (4) and (5) results in a system in the unknown coefficients A_n, B_n.

TABLE 2. Values of the Critical Parameters for Vibrational Instability

λ_+	Pr	γ_*	ω_{+*}	R_*
	1	3,81 (3,37)	6,51 (6,47)	860,06 (863,38)
0,1	10	6,39 (6,21)	66,2 (63,32)	215 (216,27)
	100	8,76 (7,52)	78,7 (77,3)	132 (135,2)

The boundary conditions (10) yield a system of homogeneous algebraic equations in the unknowns $D^2W(1)$, $D^3W(0)$, $\Theta(1)$, and $\Theta(0)$. The eigenvalue equation is obtained from the solvability condition

$$M = \frac{\begin{vmatrix} C_1^{(0)} & (C_3^{(0)}+1/2) & \left(C_3^{(0)}+\dfrac{A+1}{2}\right) \\ C_1^{(1)} & \left(C_3^{(0)}+\dfrac{B+1}{2}\right) & (C_3^{(0)}+1/2) \\ C_2^{(0)} & -C_1^{(0)}R\gamma^2 & C_1^{(0)}R\gamma^2 \end{vmatrix}}{\gamma^2\psi(\sigma)\begin{vmatrix} C_1^{(0)} & C_1^{(1)} & \left(C_3^{(0)}+\dfrac{A+1}{2}\right) \\ C_1^{(1)} & C_1^{(0)} & (C_3^{(1)}+1/2) \\ C_2^{(0)} & C_2^{(0)} & -C_1^{(0)}R\gamma^2 \end{vmatrix}} \qquad (15)$$

Here we used the notation

$$C_1^{(m)} = \sum_{n=1}^{\infty}(-1)^{mn}\frac{N_n^2}{X_n\,[\sigma\,\mathrm{Pr}^{-1}+\psi(\sigma)\,X_n](X_n+\sigma)-R\gamma^2} \quad (m=0,\ 1),$$

$$C_2^{(m)} = \sum_{n=1}^{\infty}(-1)^{mn}\frac{N_n^2\,(X_n+\sigma)}{X_n\,[\sigma\mathrm{Pr}^{-1}+\psi(\sigma)\,X_n](X_n+\sigma)-R\gamma^2} \quad (m=0,\ 1), \qquad (16)$$

$$C_3^{(m)} = \sum_{n=1}^{\infty}(-1)^{mn}\frac{X_n\,(\sigma+\gamma^2)[\sigma\,\mathrm{Pr}^{-1}+\psi(\sigma)X_n]-R\gamma^2}{X_n\,[\sigma\,\mathrm{Pr}^{-1}+\psi(\sigma)\,X_n](X_n+\sigma)-R\gamma^2} \quad (m=0,\ 1),$$

$$N_n = n\pi,\quad X_n = N_n^2+\gamma^2.$$

The numerical realization of the results (15) were produced on a "Minsk-222" electronic computer. Certain particular cases were examined.

MONOTONIC INSTABILITY

The expression (14) can then be written as

$$M = M(R,\ \gamma,\ A,\ B). \qquad (17)$$

The magnitudes of the critical Rayleigh and Marangoni numbers were calculated for two boundary conditions: $A=0$, $A=\infty$.

Setting $R = 0$ in (17), the critical Marangoni numbers can be obtained, which govern the stability threshold because of the action of only thermocapillary forces. When $M = 0$, an instability originates under the effect of only lift forces.

Values of the critical Marangoni and Rayleigh numbers, obtained for the cases $A = 0$ and $A = \infty$, are presented in Table 1. It is seen from the table that R_* and M_* grow with the rise in B. A more unstable situation corresponds to the case when $B = 0$. Then R_* has a finite value with the rise in B while M_* tends to infinity. This tendency is explained in the following manner. The increase in B from 0 to ∞ denotes a change in the thermal boundary conditions from $D\Theta = 0$ to $\Theta = 0$. Therefore, when B is small there is still great freedom for the development of temperature perturbations while for large B the temperature perturbations

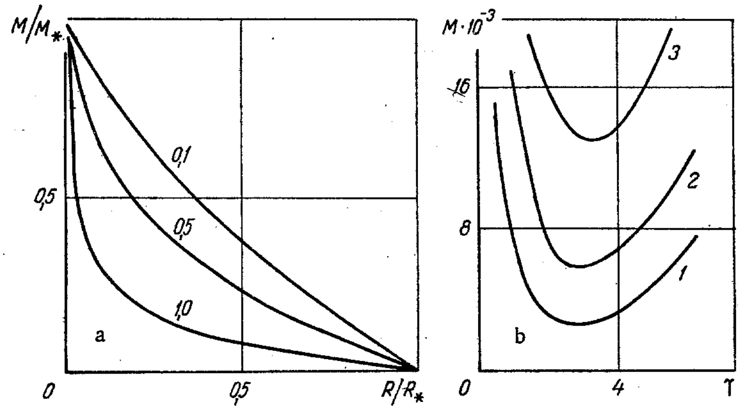

Fig. 2. Neutral stability curves ($B = \infty$, $Pr = 10^3$, $A = \infty$) (numbers at the curves are values of the elasticity parameter λ_+) (a) and influence of the boundary conditions on the neutral stability curve [$R = 0$, $Pr = 10^3$, $\lambda_+ = 1.0$. Curve 1) $A = 0$, $B = 0$; curve 2) $A = \infty$, $B = 0$; curve 3) $A = 0$, $B = 10$] (b).

damp out. In this case the influence of the thermocapillary effect is considerably less than the influence of the lift forces on the origination of a convective instability.

Values of the critical parameters obtained for certain particular cases are in good agreement with the results in [4, 5, 7-10].

VIBRATIONAL INSTABILITY

The eigenvalue equation (15) generally yields a complex Marangoni number

$$M = M_1(\gamma, R, \mathrm{Pr}, A, B, \omega, \lambda) + i\omega M_2(\gamma, R, \mathrm{Pr}, A, B, \omega, \lambda), \tag{18}$$

where M_1, M_2 are real functions and ω is the frequency of vibration. Since M should be real, the following relationships are valid:

$$\omega = 0 \quad \text{or} \quad M_2 = 0. \tag{19}$$

It has been shown in [7, 8] for the case $A = \infty$ that no real value of ω exists which satisfies $M_2 = 0$. For $A = 0$ an analogous result is obtained. However, when $B = 0$ and $Pr = 0$, there is a real value of ω which satisfies the condition $M_2 = 0$. The values of ω are hence real only as $\gamma \to 0$ and $R > 6720$. The neutral vibrational instability curve however lies in an unstable domain for stationary convection [7, 8, 11].

To determine the critical parameters governing the stability threshold, we used an integral model of a viscoelastic fluid [1]:

$$\psi(\sigma)\eta_0 = \int_0^\infty N(\tau)\exp(-\sigma\tau)\,d\tau = \eta(\sigma). \tag{20}$$

For $\sigma = 0$, $\eta_0 = \eta(0)$ is the greatest Newtonian viscosity. We should set $\sigma = i\omega$ on the threshold for the origination of a vibrational instability. Then

$$\eta(i\omega) = \eta_1(\omega) - i\eta_2(\omega). \tag{21}$$

The known relationships [12] were used for the complex viscosity components $\eta(i\omega)$. It was assumed that the dimensionless quantities satisfy the relationships $\lambda_+\omega_+ = \lambda\omega$, where λ is the fluid relaxation time, and $\lambda_+ = \lambda\varkappa/d^2$ is the elasticity parameter.

For $B = \infty$, $A = \infty$, the problem about the origination of a vibrational instability is obtained for perfectly heat conducting nonsymmetric boundaries. Then the dependence for neutral stability degenerates into the form

$$\sum_{n=1}^{\infty}(-1)^{mn}\frac{(X_n + i\omega)N_n^2}{X_n[i\omega\,\mathrm{Pr} + \eta(i\omega)/\eta_0\cdot X_n](X_n + i\omega) - R\gamma^2} = 0. \tag{22}$$

Results calculated by means of this formula are presented in Table 2. For $\omega = 0$ values are obtained for the critical Rayleigh number $R_* = 1108.8$ at $\gamma^* = 2.71$, which almost agrees with results known in the literature.

We solved a problem with nonsymmetric boundary conditions for a viscoelastic fluid separately. The Galerkin method was applied, where the following approximation was used:

$$W(z) = \sum_{m=1}^{\infty} a_m W_m(z), \quad \Theta(z) = \sum_{m=1}^{\infty} b_m \Theta_m(z), \tag{23}$$

where a_m, b_m are unknown coefficients.

The basis functions satisfying the boundary conditions (7) and (8) are odd:

$$W(z) = \frac{\mathrm{sh}(\mu_m z)}{\mathrm{sh}(\mu_m/2)} - \frac{\sin(\mu_m z)}{\sin(\mu_m/2)}, \tag{24}$$

where μ_m are positive roots of the equation

$$\mathrm{cth}\,\frac{\mu_m}{2} - \mathrm{ctg}\,\frac{\mu_m}{2} = 0. \tag{25}$$

The numerical realization on an electronic computer was carried out for $m = 2$. Values of the critical parameters are presented in parentheses in Table 2. The agreement of the data indicates reliability of the results obtained from (15). The neutral stability curve is presented in Fig. 2a for different values of the elasticity parameter λ_+.

The neutral stability curve stands off from the line $R/R_* + M/M_* = 1$ as the fluid elasticity increases. As in the monotonic instability case, it can be expected that the straight line $R/R_* + M/M_* = 1$ will correspond to the case when the connection between the two reasons for the origination of convection (lift and thermocapillary forces) is perfect, while two lines $R/R_* = 1$, $M/M_* = 1$ show the noninterrelated action of two forces.

Therefore, the connection between the two reasons for the origination of convection becomes weak as λ_+ increases. Convection can occur only because of the lift or thermocapillary forces.

The influence of the Biot number on the relative location of the stability curves is presented in Fig. 2b. The value of the critical Marangoni number drops with the rise in B.

The influence of the lower boundary is analogous to the monotonic instability case.

Therefore, the fluid elasticity exerts a destabilizing effect during the action of the lift and thermocapillary forces.

NOTATION

T, T_1, T_1°, T_2°, temperature in the fluid, the plate, in the lower plate surface, and the surrounding medium, respectively; d, d_1, fluid layer and plate thicknesses; α, coefficient of heat elimination; $\varkappa$, $\varkappa_1$, coefficient of fluid and plate heat conductivity; z, running vertical coordinate; β, temperature gradient in the fluid; W, Θ, velocity and temperature perturbation amplitudes in the fluid; γ, wave number; χ, thermal diffusivity coefficient of the fluid; δ, coefficient of thermal expansion of the fluid; σ, perturbation decrement; λ_+, elasticity parameter; Θ_1, amplitude of temperature perturbation in the plate; s, coefficient of surface tension; C, arbitrary constant; ω, vibration frequency; M_1, M_2, real functions; η_0, greatest Newtonian viscosity; $N(\tau)$, relaxation time spectrum; τ, time; $\eta(i\omega)$, complex viscosity; $\eta_1(\omega)$, $\eta_2(\omega)$, components of the complex viscosity; λ, relaxation time; a_m, b_m, C_1, coefficients; γ_*, M_*, R_*, critical values of the wave number, the Marangoni number, and the Rayleigh number; $B = \alpha d/\varkappa$, Biot number; $\underline{M} = s\beta d^2/\varkappa\eta_0$, Marangoni number; $R = \lambda\beta\nu d^4/\nu\chi$, Rayleigh number; $Pr = \nu/\chi$, Prandtl number; $\bar{\varkappa} = \varkappa/d_1$.

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SOLUTION OF A PLANE STEFAN PROBLEM FOR A HALF-SPACE BY THE METHOD OF DEGENERATE HYPERGEOMETRIC TRANSFORMATIONS

M. N. Shafeev

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A method is given for constructing the analytic solution of a plane nonstationary Stefan problem.

Analytical methods of solving multidimensional nonstationary Stefan problems have only started to be produced. Methods existing earlier for the solution of such problems ([1, 2], etc.) were quite approximate in nature. The general solution of a quasistationary plane Stefan problem is obtained in [3]. An analytical method of solving a nonstationary plane Stefan problem is proposed in this paper for a half-space in application to the process of freezing the ground bounded on one side by a plane and extending without limit to the other side.

Let us consider the problem on the dynamics of the freezing and cooling zones (zones I and II) of ground under a plane source of cold located on the surface of a semiinfinite medium (ground) (Fig. 1). The general formulation of such a problem with two moving boundaries is described by the following system of equations and boundary conditions:

$$\frac{\partial t_k(x_1, x_2, \tau)}{\partial \tau} = a_k \sum_{i=1}^{2} \frac{\partial^2 t_k(x_1, x_2, \tau)}{\partial x_i^2}, \quad k=1, 2, \tag{1}$$

$$\text{for } k=1, \ (x_1, x_2) \in D_{x,1} = \{|x_1| < \xi_1(\tau),\ 0 < x_2 < \xi(x_1, \tau)\};$$

$$\text{for } k=2, \ (x_1, x_2) \in D_{x,2} = D_x^{(1)} + D_x^{(2)};\ D_x^{(1)} = \{|x_1| \leqslant \xi_1(\tau),$$

$$\xi(x_1, \tau) < x_2 < \nu(x_1, \tau)\};\ D_x^{(2)} = \{|x_1| \geqslant \xi_1(\tau),\ |x_1| < \nu_1(\tau);$$

$$0 < x_2 < \nu(x_1, \tau)\};\quad \tau > 0;$$

$$t_k(x_1, x_2, 0) = f_k(x_1, x_2); \tag{2}$$

$$t_1(x_1, 0, \tau) = \varphi_1(x_1, \tau) \quad \text{for} \quad |x_1| \leqslant \xi_1(\tau); \tag{3}$$

$$\text{on } S_{x,0} = \{x_2 = 0,\ |x_1| \geqslant \xi_1(\tau),\ |x_1| \leqslant \nu_1(\tau)\}$$

$$t_2(x_1, x_2, \tau) = \varphi_2(x_1, \tau); \tag{4}$$

$$\text{on } S_{\tau,1} = \{x_2 = \xi(x_1, \tau),\ |x_1| \leqslant \xi_1(\tau)\}$$

$$t_k(x_1, x_2, \tau) = 0 \tag{5}$$

and

$$\sum_{i=1}^{2} \left(\lambda_1 \frac{\partial t_1(x_1, x_2, \tau)}{\partial x_i} - \lambda_2 \frac{\partial t_2(x_1, x_2, \tau)}{\partial x_i}\right) l_i = A\ \frac{\partial \xi(x_1, \tau)}{\partial \tau}; \tag{6}$$

$$\text{on } S_{\tau,2} = \{x_2 = \nu(x_1, \tau),\ |x_1| \leqslant \nu_1(\tau)\}$$

$$t_2(x_1, x_2, \tau) = f_2(x_1, x_2), \tag{7}$$

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